Methods for Evaluating Difficult Integrals

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Series Methods

Let's start with a guiding example. Suppose we wanted to evaluate

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx.$$

Standard integration tricks fail here, so let's try somethig different. Notice that

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{n+1}\,dx.$$

It would be easy to integrate each term of the series and sum the integrals up, but that's not always allowed. Here's a theorem that lets us proceed.

Theorem (Monotone convergence theorem)**.** *Suppose that on the interval (a, b) the functions $f_1, f_2, f_3, \ldots$ are all positive (that is, $f_k(x) \geq 0$ at every x). Then*

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx,$$

even if both sides are infinite. The equation is also true if the functions are all negative.

Using this theorem, let's finish evaluating our integral. On the interval $(0, 1)$ each function $x^n/(n+1)$ is positive, so

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{n+1}\,dx = -\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = -\frac{\pi^2}{6}.$$

Our method worked and we evaluated the integral. But this is important: **You can't always swap sums and integrals!** Be sure to use the above theorem whenever you want to swap.

Disclaimers aside, the method is cool enough to merit more examples.

Example 1. Let's evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

For this one we'll turn the method on its head. Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \sum_{n=1}^{\infty} \int_0^{1/2} x^n\,dx$$

Since $x^n \geq 0$ on $(0, 1/2)$,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \int_0^{1/2} \sum_{n=1}^{\infty} x^n\,dx = \int_0^{1/2} \frac{x}{1-x}\,dx = -\ln(1-x) - x\Big|_0^{1/2} = \ln 2 - \frac{1}{2}.$$

$\square$

Example 2. Let's evaluate

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx.$$

This one is forbidding indeed. The secret is the denominator; it looks like the geometric series. However, you can only expand $1/(1-t)$ if $|t| < 1$, so we rewrite:

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \int_0^\infty x^3 e^{-x} \frac{1}{1-e^{-x}}\,dx = \int_0^\infty \sum_{n=0}^\infty x^3 e^{-x} e^{-nx}\,dx.$$

Once again, all terms are positive, so

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \sum_{n=0}^\infty \int_0^\infty x^3 e^{-x} e^{-nx}\,dx = \sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx}\,dx.$$

To do the integral, a substitution makes things more managable. Let $u = nx$:

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \sum_{n=1}^\infty \left(\frac{1}{n^4}\int_0^\infty u^3 e^{-u}\,dx\right) = \left(\sum_{n=1}^\infty \frac{1}{n^4}\right)\left(\int_0^\infty u^3 e^{-u}\,du\right).$$

The problem has reduced to two separate parts. Fortunately, we know that

$$\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \text{and} \qquad \int_0^\infty u^k e^{-u}\,du = k!,$$

so our integral is

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \left(\frac{\pi^4}{90}\right)(3!) = \frac{\pi^4}{15}.$$

Whew! □

Multivariable Methods

There are many small tricks one can use when working with multiple integrals. For example, expanding in a series gives

$$\sum_{n=1}^\infty \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{dx\,dy}{1-xy}.$$

Rotating the unit square $[0,1]^2$ by 45° and changing to polar coordinates allows you to evaluate the integral—and thus the sum! We won't deal with such difficult things; there are only two methods we'll discuss.

The first method we've seen before; it's only mentioned here for review.

Example 1. Let's evaluate

$$I = \int_{-\infty}^\infty e^{-x^2}\,dx.$$

By changing to polar coordinates,

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\,dy\right) = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^\infty re^{-r^2}\,dr\,d\theta = \pi.$$

Thus $I = \sqrt{\pi}$. □

The other method is unusual, but well-known enough to have a person's name attached.

Example 2. Let's evaluate the Frullani integral

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan x}{x}\,dx.$$

This integral looks frightening to a new calculus student—and it should. The trick here is that we can view the integral as a double integral that was partially evaluated, and poorly at that. The other order of integration is much nicer.

$$\begin{aligned}
\int_0^\infty \frac{\arctan(\pi x) - \arctan x}{x}\, dx &= \int_0^\infty \left[\frac{\arctan(xy)}{x}\right]_1^\pi dx \\
&= \int_0^\infty \int_1^\pi \frac{1}{1+x^2y^2}\, dy\, dx \\
&= \int_1^\pi \int_0^\infty \frac{1}{1+x^2y^2}\, dx\, dy \\
&= \int_1^\pi \left[\frac{\arctan(xy)}{y}\right]_0^\infty dy \\
&= \int_1^\pi \frac{\pi}{2y}\, dy \\
&= \frac{\pi \ln \pi}{2}.
\end{aligned}$$

The problem appeared on the Putnam mathematical competition years ago. Such tricky integrals often do.

It's worth mentioning another way to evaluate the integral. Even though the theorem carries Liebniz's name, the method is attributed to American physicist Richard Feynman for using it so effectively. Define

$$I(a) = \int_0^\infty \frac{\arctan(ax) - \arctan x}{x}\, dx.$$

We want $I(\pi)$. Take the derivative with respect to a:

$$I'(a) = \int_0^\infty \frac{dx}{1+a^2x^2} = \frac{\pi}{2a}.$$

Note that $I(1) = 0$. Thus

$$I(\pi) = I(1) + \int_1^\pi I'(a)\, da = 0 + \int_1^\pi \frac{\pi}{2a}\, da = \frac{\pi \ln \pi}{2}.$$

□

Miscellaneous Tricks

Finally, we have a couple of examples that don't use any of the machinery we've developed. They're just neat problems.

Example 1. Let's evaluate, for $a > 0$,

$$I(a) = \int_0^\infty \frac{\ln x}{x^2 + a^2}\, dx.$$

First a substitution $x = ay$ gives us

$$I(a) = \frac{1}{a}\int_0^\infty \frac{\ln(ay)}{y^2+1}\, dy = \frac{1}{a}\int_0^\infty \frac{\ln a}{y^2+1}\, dy + \frac{1}{a}\int_0^\infty \frac{\ln y}{y^2+1}\, dy = \frac{\pi \ln a}{2a} + \frac{1}{a} I(1).$$

Now we have to evaluate the original integral with $a = 1$. For this we use a trick I like to call a "bound flip." Sometimes reversing the bounds in a simple (but nontrivial) way can show us something interesting about the original integral. In this case, let's set $u = 1/y$, so that

$$I(1) = \int_\infty^0 \frac{\ln(1/u)}{(1/u)^2+1} \frac{du}{(-u^2)} = \int_0^\infty -\frac{\ln(u)}{u^2+1}\, dy = -I(1).$$

But this means that $I(1) = -I(1)$, so we conclude $I(1) = 0$. The original integral is $I(a) = (\pi \ln a)/(2a)$. □

A bound flip also solved a mathematical competition problem years ago.

Example 2. Let's evaluate

$$I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

Setting $x = \pi/2 - y$ gives $\tan x = \cot y = 1/\tan y$, so

$$I = \int_{\pi/2}^{0} \frac{-dy}{1 + (\cot y)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan y)^{\sqrt{2}} dy}{1 + (\tan y)^{\sqrt{2}}}.$$

If we add the original integral...

$$I + I = \int_0^{\pi/2} \left(\frac{1}{1 + (\tan x)^{\sqrt{2}}} + \frac{(\tan x)^{\sqrt{2}}}{1 + (\tan x)^{\sqrt{2}}} \right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Thus $I = \pi/4$. $\square$